Fixed point theorem for three mappings on three complete metric spaces, using implicit relations

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Abstract. A fixed point theorem in three metric spaces is proved. This result extends the results obtained in [3] from two metric spaces to three metric spaces. It generalizes the results obtained in [6,7,8]. A several corollaries are obtained according as the forms of implicit functions.

1. Introduction

In [6], [7] and [3] the following theorems are proved:

Theorem 1 (*Nung*) [6] *Let* $(X,d),(Y,\rho)$ *and* (Z,σ) *be complete metric spaces and suppose* T *is a continuous mapping of* X *into* Y, S *is a continuous mapping of* Y *into* Z *and* R *is a continuous mapping of* Z *into* X *satisfying the inequalities*

$$d(RSTx,RSy) \le c\max\{d(x,RSy),d(x,RSTx),\rho(y,Tx),\sigma(Sy,STx)\}$$

$$\rho(TRSy,TRz) \le c\max\{\rho(y,TRz),\rho(y,TRSy),\sigma(z,Sy),d(Rz,RSy)\}$$

$$\sigma(STRz,STx) \le c\max\{\sigma(z,STx),\sigma(z,STRz),d(x,Rz),\rho(Tx,TRz)\}$$

for all x in X, y in Y and z in Z, where $0 \le c < 1$. Then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu = v, Sv = w$ and $Rw = u$.

Theorem 2 (*Jain et.al.*)[7] *Let* $(X,d),(Y,\rho)$ *and* (Z,σ) *be complete metric spaces and suppose* T *is a mapping of* X *into* Y, S *is a mapping of* Y *into* Z *and* R *is a mapping of* Z *into* X *satisfying the inequalities*

$$d^2(RSy,RSTx) \le c\max\ \{d(x,RSy)\rho(y,Tx),\rho(y,Tx)d(x,RSTx), d(x,RSTx)\sigma(Sy,STx),\sigma(Sy,STx)d(x,RSy)\}$$

$$\rho^2(TRz,TRSy) \le c\max\ \{\rho(y,TRz)\sigma(z,Sy),\sigma(z,Sy)\rho(y,TRSy), \rho(y,TRSy)d(Rz,RSy),d(Rz,RSy)\rho(y,TRz)\}$$

$$\sigma^2(STx,STRz) \le c\max\ \{\sigma(z,STx)d(x,Rz),d(x,Rz)\sigma(z,STRz), \sigma(z,STRz)\rho(Tx,TRz),\rho(Tx,TRz)\sigma(z,STx)\}$$

for all x in X, y in Y and z in Z, where $0 \le c < 1$. If one of the mappings R, S, T is continuous, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu = v, Sv = w$ and $Rw = u$.

Theorem 3 (Nešić)[3] *Let (X,d) and (Y,ρ) be complete metric spaces. Let T be a mapping of X into Y and S a mapping of Y into X. Denote*

$$M_1(x,y) = \{d^p(x,Sy), \rho^p(y,Tx), \rho^p(y,TSy)\}$$

and

$$M_2(x,y) = \{\rho^p(y,Tx), d^p(x,Sy), d^p(x,STx)\}$$

for all x in X, y in Y and $p = 1,2,3,\ldots$.

Let R^+ be the set of nonnegative real numbers, and let $F_i : R^+ \to R^+$ be a mapping such that $F_i(0) = 0$ and F_i is continuous at 0 for $i = 1,2$.

If T and S satisfying the inequalities

$$\rho^p(Tx,TSy) \le c_1 \max M_1(x,y) + F_1(\min M_1(x,y)),$$

$$d^p(Sy,STx) \le c_2 \max M_2(x,y) + F_2(\min M_2(x,y)),$$

for all x in X and y in Y, where $0 \le c_1, c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

2. Main results

We will prove a theorem which generalizes the Theorems Nung [6], Jain, Shrivastava and Fisher [7], Nešić [3] and extends the Theorem Nešić from two to three metric spaces. For this, we will use the implicit functions.

Let $\Phi_4^{(m)}$ be the set of continuous functions with 4 variables

$$\varphi : [0,+\infty)^4 \to [0,+\infty)$$

satisfying the properties:

φ is non descending in respect with each variable.

$\varphi(t,t,t,t) \le t^m, m \in N$.

Denote $I_4 = \{1,2,3,4\}$.

Some examples of such functions are as follows:

Example 4 $\varphi(t_1,t_2,t_3,t_4) = \max\{t_1,t_2,t_3,t_4\}$, *with* $m = 1$.

Example 5 $\varphi(t_1,t_2,t_3,t_4) = \max\{t_i t_j : i, j \in I_4\}$, *with* $m = 2$.

Example 6 $\varphi(t_1,t_2,t_3,t_4) = \max\{t_1t_2, t_2t_3, t_3t_4, t_4t_1\}$, *with* $m = 2$.

Example 7 $\varphi(t_1,t_2,t_3,t_4)=\max\{t_1^p,t_2^p,t_3^p,t_4^p\}$, *with* $m=p$.

Let Ψ_4 be the set of continuous functions with 4 variables

$$\psi:[0,+\infty)^4\to[0,+\infty)$$

satisfying the property

$$t_1t_2t_3t_4=0\Rightarrow\psi(t_1,t_2,t_3,t_4)=0.$$

Example 8

$$\psi(t_1,t_2,t_3,t_4)=\min\{t_1,t_2,t_3,t_4\}$$
$$\psi(t_1,t_2,t_3,t_4)=\min\{t_1,t_2,t_3\}$$
$$\psi(t_1,t_2,t_3,t_4)=\min\{t_1^p,t_2^p,t_3^p,t_4^p\},etc.$$

Let $\mathcal{F}$ be the set of continuous functions

$$F:[0,+\infty)\to[0,+\infty)$$

with $F(0)=0$ (For example $F(t)=t^k,k>0$).

Theorem 9 *Let $(X,d),(Y,\rho)$ and (Z,σ) be complete metric spaces and suppose T is a mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X, such that at least one of them is a continuous mapping. Let $\varphi_i\in\Phi_4^{(m)},\psi_i\in\Psi_4,F_i\in\mathcal{F}$ for $i=1,2,3$. If there exists $q\in[0,1)$ and the following inequalities hold*

(1) $$d^m(RSy,RSTx)\le q\varphi_1(d(x,RSy),d(x,RSTx),\rho(y,Tx),\sigma(Sy,STx))+$$
$$+F_1(\psi_1(d(x,RSy),d(x,RSTx),\rho(y,Tx),\sigma(Sy,STx)).$$

(2) $$\rho^m(TRz,TRSy)\le q\varphi_2(\rho(y,TRz),\rho(y,TRSy),\sigma(z,Sy),d(Rz,RSy))+$$
$$+F_2(\psi_2(\rho(y,TRz),\rho(y,TRSy),\sigma(z,Sy),d(Rz,RSy)).$$

(3) $$\sigma^m(STx,STRz)\le q\varphi_3(\sigma(z,STx),\sigma(z,STRz),d(x,Rz),\rho(Tx,TRz))+$$
$$+F_3(\psi_3(\sigma(z,STx),\sigma(z,STRz),d(x,Rz),\rho(Tx,TRz))$$

for all $x\in X,y\in Y$ and $z\in Z$, then RST has a unique fixed point $\alpha\in X$, TRS has a unique fixed point $\beta\in Y$ and STR has a unique fixed point $\gamma\in Z$. Further, $T\alpha=\beta,S\beta=\gamma$ and $R\gamma=\alpha$.

Let $x_0\in X$ be an arbitrary point. We define the sequences $(x_n),(y_n)$ and (z_n) in X,Y and Z respectively as follows:

$$x_n=(RST)^n x_0,y_n=Tx_{n-1},z_n=Sy_n,n=1,2,...$$

Denote

$$d_n=d(x_n,x_{n+1}),\rho_n=\rho(y_n,y_{n+1}),\sigma_n=\sigma(z_n,z_{n+1}),n=1,2,...$$

By the inequality (2), for $y=y_n$ and $z=z_{n-1}$ we get:

$$\rho^m(y_n, y_{n+1}) \le q\varphi_2(\rho(y_n, y_n), \rho(y_n, y_{n+1}), \sigma(z_{n-1}, z_n), d(x_{n-1}, x_n)) +$$
$$+F_2(\psi_2(\rho(y_n, y_n), \rho(y_n, y_{n+1}), \sigma(z_{n-1}, z_n), d(x_{n-1}, x_n)).$$

or

$$\rho_n^m \le q\varphi_2(0, \rho_n, \sigma_{n-1}, d_{n-1}) + F_2(\psi_2(0, \rho_n, \sigma_{n-1}, d_{n-1})) =$$
$$= q\varphi_2(0, \rho_n, \sigma_{n-1}, d_{n-1}) \qquad (4)$$

For the coordinates of the point $(0, \rho_n, \sigma_{n-1}, d_{n-1})$ we have:

$$\rho_n \le \max\{d_{n-1}, \sigma_{n-1}\}, \forall n \in N \ (5)$$

because, in case that $\rho_n > \max\{d_{n-1}, \sigma_{n-1}\}$ for some n, if we replace the coordinates with ρ_n and apply the property (b) of φ_2 we get:

$$\rho_n^m \le q\varphi_2(\rho_n, \rho_n, \rho_n, \rho_n) \le q\rho_n^m.$$

This is impossible since $0 \le q < 1$.

By the inequalities (4), (5) and properties of φ_2 we get:

$$\rho_n^m \le q\varphi_2(\max\{d_{n-1}, \sigma_{n-1}\}, \max\{d_{n-1}, \sigma_{n-1}\}, \max\{d_{n-1}, \sigma_{n-1}\}, \max\{d_{n-1}, \sigma_{n-1}\}) \le$$
$$\le q \max\{d_{n-1}^m, \sigma_{n-1}^m\}.$$

Thus

$$\rho_n \le \sqrt[m]{q} \max\{d_{n-1}, \sigma_{n-1}\} \ (6)$$

By the inequality (4), for $x = x_{n-1}$ and $z = z_n$ we get:

$$\sigma^m(z_n, z_{n+1}) \le q\varphi_3(\sigma(z_n, z_n), \sigma(z_n, z_{n+1}), d(x_{n-1}, x_n), \rho(y_n, y_{n+1})) +$$
$$+F_3(\psi_3(\sigma(z_n, z_n), \sigma(z_n, z_{n+1}), d(x_{n-1}, x_n), \rho(y_n, y_{n+1})))$$

or

$$\sigma_n^m \le q\varphi_3(0, \sigma_n, d_{n-1}, \rho_n) + F_3(0) =$$
$$= q\varphi_3(0, \sigma_n, d_{n-1}, \rho_n) \qquad (7)$$

In similar way, we get:

$$\sigma_n^m \le q \max\{d_{n-1}^m, \rho_n^m\}, \forall n \in N.$$

By this inequality and (6) we get:

$$\sigma_n \le \sqrt[m]{q} \max\{d_{n-1}, \sigma_{n-1}\}, \forall n \in N \ (8)$$

By (1) for $x = x_n$ and $y = y_n$ we get:

$$d^m(x_n, x_{n+1}) \le q\varphi_1(d(x_n, x_n), d(x_n, x_{n+1}), \rho(y_n, y_{n+1}), \sigma(z_n, z_{n+1})) +$$
$$+F_1(\psi_1(d(x_n, x_n), d(x_n, x_{n+1}), \rho(y_n, y_{n+1}), \sigma(z_n, z_{n+1})))$$

or

$$d_n^m \le q\varphi_1(0, d_n, \rho_n, \sigma_n) + F(0) =$$
$$= q\varphi_1(0, d_n, \rho_n, \sigma_n) \qquad (9)$$

For the same reasons we used to (5), for the coordinates of the point $(0, d_n, \rho_n, \sigma_n)$ we have:

$$d_n \le \max\{\rho_n, \sigma_n\}, \forall n \in N.$$

Applying to (9) the properties of φ_1 and the inequalities (6), (8) we get:

$$d_n \le \sqrt[m]{q}\max\{\rho_n, \sigma_n\} \le \sqrt[m]{q}(\sqrt[m]{q}\max\{d_{n-1}, \sigma_{n-1}\}) =$$
$$= \sqrt[m]{q}(\sqrt[m]{q})\max\{d_{n-1}, \sigma_{n-1}\} \le \sqrt[m]{q}\max\{d_{n-1}, \sigma_{n-1}\}$$

or

$$d_n \le \sqrt[m]{q}\max\{d_{n-1}, \sigma_{n-1}\} \ (10)$$

By the inequalities (6), (8) and (10), using the mathematical induction, we get:

$$d(x_n, x_{n+1}) \le r^{n-1}\max\{d(x_1, x_2), \sigma(z_1, z_2)\}$$
$$\rho(y_n, y_{n+1}) \le r^{n-1}\max\{d(x_1, x_2), \sigma(z_1, z_2)\}$$
$$\sigma(z_n, z_{n+1}) \le r^{n-1}\max\{d(x_1, x_2), \sigma(z_1, z_2)\}$$

where $\sqrt[m]{q} = r < 1$.

Thus the sequences $(x_n), (y_n)$ and (z_n) are Cauchy sequences. Since the metric spaces $(X, d), (Y, \rho)$ and (Z, σ) are complete metric spaces we have:

$$\lim_{n\to\infty} x_n = \alpha \in X, \lim_{n\to\infty} y_n = \beta \in Y, \lim_{n\to\infty} z_n = \gamma \in Z.$$

Assume that S is a continuous mapping. Then by

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} z_n.$$

it follows

$$S\beta = \gamma. \ (11)$$

By (1), for $y = \beta$ and $x = x_n$ we get:

$$d^m(RS\beta, x_{n+1}) \le q\varphi_1(d(x_n, RS\beta), d(x_n, x_{n+1}), \rho(\beta, y_{n+1}), \sigma(\gamma, S\beta)) +$$
$$+F_1(\psi_1(d(x_n, RS\beta), d(x_n, x_{n+1}), \rho(\beta, y_{n+1}), \sigma(\gamma, S\beta))).$$

By this inequality and (11) we get:

$$d^m(RS\beta, x_{n+1}) \le q\varphi_1(d(x_n, RS\beta), d(x_n, x_{n+1}), \rho(\beta, y_{n+1}), 0) +$$
$$+F_1(0).$$

Letting n tend to infinity, we get

$$d^m(RS\beta, \alpha) \le q\varphi_1(d(RS\beta, \alpha), 0, 0, 0) \le qd^m(RS\beta, \alpha)$$

or

$$d(RS\beta, \alpha) = 0 \Leftrightarrow RS\beta = \alpha. \ (12)$$

By (2), for $z = S\beta$ and $y = y_n$ we get:

$$\rho^m(TRS\beta, y_{n+1}) \le q\varphi_2(\rho(y_n, TRS\beta), \rho(y_n, y_{n+1}), \sigma(S\beta, z_n), d(x_n, RS\beta)) +$$
$$+F_2(\psi_2(\rho(y_n, TRS\beta), \rho(y_n, y_{n+1}), \sigma(S\beta, z_n), d(x_n, RS\beta))).$$

Letting n tend to infinity and using (11), (12) we get:

$$\rho^m(TRS\beta, \beta) \le q\varphi_2(\rho(\beta, TRS\beta), 0, 0, 0) + F(0).$$

or

$$\rho^m(TRS\beta, \beta) \le q\rho^m(\beta, TRS\beta) \Leftrightarrow TRS\beta = \alpha. \quad (13)$$

By (11), (12) and (13) it follows:

$$TRS\beta = TR\gamma = T\alpha = \beta$$
$$STR\gamma = ST\alpha = S\beta = \gamma$$
$$RST\alpha = RS\beta = \beta\gamma = \alpha$$

Thus, we proved that the points α, β, γ are fixed points of RST, TRS and STR respectively.

In the same conclusion we would arrive if one of the mappings R or T would be continuous.

Let we prove now the iniquity of the fixed points α, β and γ.

Assume that there is α' a fixed point of RST different from α.

By (1) for $x = \alpha'$ and $y = T\alpha$ we get:

$$d^m(\alpha, \alpha') = d^m(RST\alpha, RST\alpha') \le$$
$$\le q\varphi_1(d(\alpha', RST\alpha), d(\alpha', RST\alpha'), \rho(T\alpha, T\alpha'), \sigma(ST\alpha, ST\alpha')) +$$
$$+F_1(\psi_1(d(\alpha', RST\alpha), d(\alpha', RST\alpha'), \rho(T\alpha, T\alpha'), \sigma(ST\alpha, ST\alpha'))) =$$
$$= q\varphi_1(d(\alpha', \alpha), 0, \rho(T\alpha, T\alpha'), \sigma(ST\alpha, ST\alpha')) + F(0) \le$$
$$\le q\max\{d^m(\alpha', \alpha), \rho^m(T\alpha, T\alpha'), \sigma^m(ST\alpha, ST\alpha')\}$$

or

$$d^m(\alpha, \alpha') = q\max A \quad (14)$$

where $A = \{d^m(\alpha', \alpha); \rho^m(T\alpha, T\alpha'); \sigma^m(ST\alpha, ST\alpha')\}$.

We distinguish the following three cases:

Case I: If $\max A = d^m(\alpha', \alpha)$, then the inequality (14) implies

$$d^m(\alpha, \alpha') \le qd^m(\alpha', \alpha) \Leftrightarrow \alpha' = \alpha.$$

Case II: If $\max A = \rho^m(T\alpha, T\alpha')$, then the inequality (14) implies

$$d^m(\alpha, \alpha') \le q\rho^m(T\alpha, T\alpha') \quad (15)$$

Continuing our argumentation for the Case 2, by (2) for $z = ST\alpha$ and $y = T\alpha'$ we have:

$$\rho^m(T\alpha,T\alpha') = \rho^m(TRST\alpha,TRST\alpha') \le$$
$$\le q\varphi_2(\rho(T\alpha',TRST\alpha),\rho(T\alpha',TRST\alpha'),\sigma(ST\alpha,ST\alpha'),d(RST\alpha',RST\alpha))$$
$$+F_2(\psi_2(\rho(T\alpha',TRST\alpha),\rho(T\alpha',TRST\alpha'),\sigma(ST\alpha,ST\alpha'),d(RST\alpha',RST\alpha))) =$$
$$= q\varphi_2(\rho(T\alpha',T\alpha),0,\sigma(ST\alpha,ST\alpha'),d_1(\alpha,\alpha'))+F(0) =$$
$$\le q\max A \tag{16}$$

Since in Case II, $\max A = \rho^m(T\alpha,T\alpha')$, by (16) it follows

$$\rho^m(T\alpha,T\alpha') \le q\rho^m(T\alpha,T\alpha')$$

or

$$\rho(T\alpha,T\alpha') = 0.$$

By (15), it follows $d(\alpha,\alpha') = 0$.

Case III: If $\max A = \sigma^m(ST\alpha,ST\alpha')$, then by (14) it follows

$$d^m(\alpha,\alpha') \le q\sigma^m(ST\alpha,ST\alpha') \quad (17)$$

By the inequality (3), for $x = RST\alpha, z = ST\alpha'$, in similar way we obtain:

$$\sigma^m(ST\alpha,ST\alpha') \le q\max A = q\sigma^m(ST\alpha,ST\alpha')$$

It follows

$$\sigma(ST\alpha,ST\alpha') = 0$$

and by (17) it follows

$$d(\alpha,\alpha') = 0.$$

Thus, we have again $\alpha = \alpha'$.

In the same way, it is proved the nicety of β and γ.

We emphasize the fact that it is necessary the continuity of at least one of the mappings T,S and R. The following example shows this.

Example 10 *Let* $X = Y = Z = [0,1]; d = \rho = \sigma$ *such that* $d(x,y) = |x-y|, \forall x,y \in [0,1]$. *We consider the mappings* $T,S,R:[0,1] \to [0,1]$ *such that*

$$Tx = Rx = Sx = \begin{cases} 1 & \text{for } x = 0 \\ \frac{x}{2} & \text{for } x \in (0,1] \end{cases}$$

We have

$$STx = RSx = TRx = \begin{cases} \frac{1}{2} & for\ x = 0 \\ \frac{x}{4} & for\ x \in (0,1] \end{cases}$$

and

$$RSTx = TRSx = STRx = \begin{cases} \frac{1}{4} & for\ x = 0 \\ \frac{x}{8} & for\ x \in (0,1] \end{cases}$$

We observe that the inequalities (1), (2) and (3) are satisfied for $\varphi_1 = \varphi_2 = \varphi_3 = \varphi \in \Phi_4^{(1)}$ with $\varphi(t_1,t_2,t_3,t_4) = \max\{t_1,t_2,t_3,t_4\}$, where $q = \frac{1}{2}$ and $F = 0$. It can be seen that none of the mappings RST, TRS, STR has a fixed point. This is because none of the mappings T, R, S is a continuous mapping.

3. Corollaries

Corollary 3.1 *Let $(X,d),(Y,\rho)$ and (Z,σ) be complete metric spaces and suppose T is a mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X, such that at least one of them is a continuous mapping. Let $F:[0,+\infty) \to [0,+\infty)$ be continuous with $F(0)=0$. If there exists $q \in [0,1)$ and $m \in N$ such that the following inequalities hold*

$$(1) \qquad d^m(RSy, RSTx) \le q\max(d(x,RSy), d(x,RSTx), \rho(y,Tx), \sigma(Sy,STx)) + \\ + F_1(\min(d(x,RSy), d(x,RSTx), \rho(y,Tx), \sigma(Sy,STx)).$$

$$(2) \qquad \rho^m(TRz, TRSy) \le q\max(\rho(y,TRz), \rho(y,TRSy), \sigma(z,Sy), d(Rz,RSy)) + \\ + F_2(\min(\rho(y,TRz), \rho(y,TRSy), \sigma(z,Sy), d(Rz,RSy)).$$

$$(3) \qquad \sigma^m(STx, STRz) \le q\max(\sigma(z,STx), \sigma(z,STRz), d(x,Rz), \rho(Tx,TRz)) + \\ + F_3(\min(\sigma(z,STx), \sigma(z,STRz), d(x,Rz), \rho(Tx,TRz))$$

for all $x \in X, y \in Y$ and $z \in Z$, then RST has a unique fixed point $\alpha \in X$, TRS has a unique fixed point $\beta \in Y$ and STR has a unique fixed point $\gamma \in Z$. Further, $T\alpha = \beta, S\beta = \gamma$ and $R\gamma = \alpha$.

The proof follows by Theorem 2.6 in the case $F_1 = F_2 = F_3 = F, \varphi_1 = \varphi_2 = \varphi_3 = \varphi \in \Phi_4^{(m)}$ such that $\varphi(t_1,t_2,t_3,t_4) = \max\{t_1^m, t_2^m, t_3^m, t_4^m\}$ and $\psi_1 = \psi_2 = \psi_3 = \psi$, where $\psi(t_1,t_2,t_3,t_4) = \min\{t_1^m, t_2^m, t_3^m, t_4^m\}$.

Corollary 3.1 extends Theorem 1.3 (Nešić [3]) from two in three metric spaces.

Corollary 3.2 *Theorem 1.1 (Nung [6]) is taken by Corollary 3.1 for* $m=1$ *and* $F=0$.

Corollary 3.3 *Theorem 1.2 (Jain et. al. [7]) is taken by Theorem 2.6 in case* $F_1 = F_2 = F_3 = 0; \varphi_1 = \varphi_2 = \varphi_3 = \varphi \in \Phi_4^{(2)}$ *such that* $\varphi(t_1,t_2,t_3,t_4) = \max\{t_1t_3, t_2t_3, t_2t_4, t_4t_1\}$.

Corollary 3.4 *Theorem Kikina (Theorem 2.1, [8]) is taken by Corollary 3.1 in case* $\varphi(t_1,t_2,t_3,t_4) = \max\{t_1^m, t_2^m, t_3^m\}$ *and* $\psi(t_1,t_2,t_3,t_4) = \min\{t_1^m, t_2^m, t_3^m\}$.

Corollary 3.5 *Let* $(X,d),(Y,\rho)$ *be complete metric spaces and suppose* T *is a mapping of* X *into* Y, S *is a mapping of* Y *into* Z. $\varphi_i \in \Phi_3, F_i \in \mathbb{F}$ *for* $i=1,2$. *If there exists* $q \in [0,1)$ *such that the following inequalities hold*

$$(1') \qquad d(Sy,STx) \le q\varphi_1(d(x,Sy), d(x,STx), \rho(y,Tx)) + F_1(\psi_1(d(x,Sy), d(x,STx), \rho(y,Tx)).$$

$$(2') \qquad \rho^m(Tx,TSy) \le q\varphi_2(\rho(y,Tx), \rho(y,TSy), d(x,Sy)) + F_2(\psi_2(\rho(y,Tx), \rho(y,TSy), d(x,Sy)).$$

for all $x \in X, y \in Y$, then ST has a unique fixed point $\alpha \in X$ and TS has a unique fixed point $\beta \in Y$. Further, $T\alpha = \beta, S\beta = \gamma$.

By Theorem 2.6, if we take: $Z = X, \sigma = d$ the mapping R as the identity mapping in X, $\varphi_i(t_1,t_2,t_3,t_4) = \varphi_i(t_1,t_2,t_3), \psi_i(t_1,t_2,t_3,t_4) = \psi_i(t_1,t_2,t_3)$, then the inequality (1) takes the form (1'), the inequality (2) takes the form (2') and the inequality (3) is always satisfied since his left side is $\sigma^m(STx,STx) = 0$. Thus, the satisfying of the conditions (1), (2) and (3) is reduced in satisfying of the conditions (1') and (2').

The mappings T and S may be not continuous, while fromt he mappings T,S and R for which we applied Theorem 2.6, the identity mapping R is continuous. This completes the proof.

We have the following corollary.

Corollary 3.6 *(Theorem Nešić [3]). Theorem 1.3 is taken by Corollary 3.5 for* $\varphi_1 = \varphi_2 = \varphi; \psi_1 = \psi_2 = \psi$ *such that* $\varphi_1(t_1,t_2,t_3) = \max\{t_1^m, t_2^m, t_3^m\}$ *and* $\psi(t_1,t_2,t_3) = \min\{t_1^m, t_2^m, t_3^m\}$.

We emphasize the fact that in the Theorem 1.3, the mappings F_1 and F_2 can be replaced by $F(t) = \max\{F_1(t), F_2(t)\}$ and c_1, c_2 can be replaced by $q = \max\{c_1, c_2\}$.

Corollary 3.7 *Theorem Popa (Theorem 2, [2]) is taken by Corollary 3.5 for*

$\varphi_1 = \varphi_2 = \varphi$ *such that* $\varphi(t_1, t_2, t_3) = \max\{t_1t_2, t_1t_3, t_2t_3\}$ *with* $m = 2$ *and* $F = 0$.

We also emphasize here that the constants c_1, c_2 can be replaced by $q = \max\{c_1, c_2\}$.

Remark. As corollaries of these results we can obtain other propositions determined by the form of implicit functions, for example Proposition Popa (Corollary 2, [2]), Theorem Fisher (Theorem 1, [1]) etc.

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